A Simple Algebraic Logrolling Model

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In the rapidly developing literature in which essentially economic tools are applied to political problems, there have been two major models of voting performance. One of the models, by all odds the most widely used, is essentially spatial. In it, individuals are assumed to have a preference mountain and to prefer the points which are closer to their optimum to points which are farther away. This model, which started as a very simple one-dimensional continuum in the work of Harold Hotelling, Duncan Black (1948), and Anthony Downs, has developed into a more complex, many dimensional model in the later work of Black (1958), Otto Davis and M. J. Hinich (1966, 1967, 1968), and Tullock. The many dimensional version of this model must be represented, of course, by some variant on the Cartesian algebra since it is not easy to represent graphically more than two dimensions on a piece of paper. In general, these models have been used mainly to demonstrate that in a two-party system, the two parties will normally have platforms that are very similar and that these will represent median preferences. The other model deals with the phenomenon of logrolling and has normally been represented by other tools, see James Buchanan and Tullock. The interrelation between these two models has been discussed in general by Davis and Hinich (1968, p. 68) and Tullock (pp. 57–61), but no very rigorous joint model exists. It is the purpose of this article to demonstrate that the two approaches are not inconsistent by presenting a spatial model which will also cover logrolling.

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I. *Three-Person Model*

Suppose that an individual must choose government policies with respect to three different issues which we shall designate A, B, and C, and that each of these issues represents a continuum, such as the appropriation for the army or the appropriation for the welfare program. This situation can be represented by a three-dimensional issue space with each of the issues representing one dimension and an individual having some point which is for him optimal, let us say [10, 10, 10]. Presumably his level of satisfaction will fall off as the actual social choice moves away from his optimum. If we assume that this fall-off is uniform in every direction, we may express his loss from not achieving his optimum by the equation:

$$(1)\quad L_A^2 = [A - 10]^2 + [B - 10]^2 + [C - 10]^2$$

If we had a number of people with varying optima in the issue space, we would be able to deduce from the resulting set of equations similar to equation (1) how they would vote on each proposition that was put before them.[1] As has been demonstrated by the spatial models so far published, except with very special distributions of optima, the outcome under simple majority voting would be some point which is approximately at the median of the entire distribution. This conclusion is readily generalizable up to any number of issues, since the Cartesian system can be applied to an issue space of any number of dimensions.

The use of perfectly spherical indiffer-

[1] See Colin Campbell and Tullock for an application of this process.

ence surfaces in this model does not appear to restrict its utility very much. In the real world, we would not anticipate such perfection, but the deviations from it would be essentially random and the law of large numbers should lead, where there are many voters, to much the same outcome as if we used our spheres. For a demonstration, see Kenneth Arrow. Systematic deviations from the spherical model, together with appropriately structured locations of the individual optima, could lead to voting cycles, and the conclusion that the median preference dominates would be undermined. It is the purpose of this article to consider an important case in which we would anticipate that the individual indifference curves would systematically vary from the spherical in a particular way, and in which we would anticipate that individual preferences would have a structure such that the combination of these two effects leads to quite different results than have customarily been dealt with by the spatial models.

If we consider those situations in the real world in which we observe logrolling and compare them with those situations where logrolling appears to be relatively unimportant, we observe immediate differences in the structure of the individual preferences. In logrolling, we observe a number of people who are highly interested in one particular project, let us say, the dredging of the James so that Richmond becomes a deep water port, and only mildly interested in other projects which, generally speaking, they oppose. The rivers and harbors area is the *locus classicus* of logrolling, but similar phenomenon will be found throughout a very large part of modern governments.

The indifference curves of the individuals engaging in logrolling are somewhat similar to those shown on Figure 1. Mr. A wants his harbor dredged at the expense of the general taxpayer and feels quite

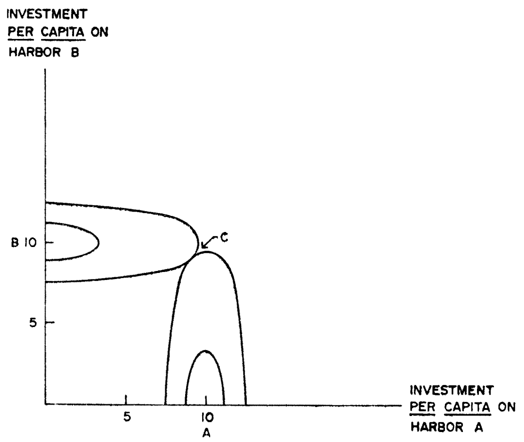

FIGURE 1

strongly about it, but he would rather not pay for dredging Mr. B's harbor. Since he is only one of many taxpayers, however, his feeling about the dredging of Mr. B's harbor is much feebler than his feeling about his own. Mr. B's feelings are the converse. If we assume that all of the citizens of the town in which Mr. A lives feel much the same as Mr. A[2] and the other citizens of the town in which Mr. B lives feel much the same as Mr. B, then logrolling becomes rational. Point *C* is better than the origin for both A and B. It is not, however, possible to represent a many dimensional logrolling process in two dimensions and, if we consider such a piece of legislation as the rivers and harbors bill, it is clear that several thousand dimensions would be necessary.

We can begin with a simple three-dimensional model using the ordinary Cartesian algebra. This model for logrolling will differ from the usual spatial model only in that the individuals are assumed to have intense preferences on certain subjects. If we assume a three-person society where

[2] This does not mean that their optima or intensities are the same as Mr. A, but simply that all of their indifference curves would have the same general shape as Mr. A's.

harbor dredging is paid for by equal per capita taxes, and that there are three harbor dredging operations contemplated (A, B, and C), then Mr. A's preferences can be represented by the first equation in Table 1. Mr. A is assumed to have his personal optimum at the point [10, 0, 0]. L_A is the "loss" he suffers if the government chooses some other point. As in the real world, he is much more interested in the dredging of his own harbor than in preventing the dredging of the other two harbors, although he doesn't like paying taxes to benefit other people. Note that Mr. A's optima includes his own per capita share in terms of tax payments for his own harbor. For simplicity we will continue to assume throughout that all expenditures on logrolled projects are paid for by a tax which is evenly divided among the taxpayers and the per capita cost is our metric on each issue dimension.

II. *Logrolling Results*

Taken on the two dimensions represented by axes A and B and holding C equal to zero, the indifference curves generated by the equations of A and B in the positive quadrant will approximate those shown in Figure 1. In three dimensions, A's indifference surfaces in the positive part of the issue space would form a quarter of a disk with its center at the point [10, 0, 0]. The other three individuals in our current simple model would have similar disks attached to the other three axes. If the voting rule is simple majority voting, and each individual votes for his preference on each of the three harbor proposals, then there will be two votes against each proposal and all will fail. L_A would equal 22.4. The individuals, however, should notice the possibility for gains from trade. If two of them could get together and vote for each other's harbor dredging project, then they can make a quite considerable gain. Bargaining difficulties in this case are apt to be minimal since each party to the bargain has the alternative of turning to the third party and hence in essence they are operating in a market type situation.[3] Thus, if we assume that there is some agreement between Mr. A and Mr. B and that they choose an equal amount of harbor dredging in each of their harbors,[4] it's fairly easy to determine the point in the issue space which would result. It is [8 1/3, 8 1/3, 0]. L_A and L_B will be 9.1,[5] very much better than the situation without the agreement. L_C, on the other hand, is now 27.2, much worse than the situation before the agreement was made. The reason, of course, is simple. Mr. C's harbor is not being dredged and he is paying taxes to dredge the other two.

This may be taken as a very simple example of the type of bargain which occurs in logrolling. In practice, things are more complicated. There are basically two types

TABLE 1

$L_A^2 = 5(A-10)^2 + B^2 + C^2$
$L_B^2 = A^2 + 5(B-10)^2 + C^2$
$L_C^2 = A^2 + B^2 + 5(C-10)^2$

[3] A modern theory of bargaining can be said to have been initiated by J. von Neumann and O. Morgenstern. For a summary of developments since publication of *Theory of Games and Economic Behavior* and some interesting experimental results in a three-person situation, see William Riker (1967).

[4] The assumption of equal division is not strictly necessary for the general conclusions reached below. The equations would also be solvable for other assumptions as to the division of the spoils between the members of the logrolling bargain.

[5] The derivation of these results is simple but not obvious. Since A is equal to B, and C is equal to 0, the first equation in Table 1 reduces to: $L_A^2 = 6A^2 - 100A + 500$. Differentiating [the variable L_A^2 with respect to A] and setting the differential equal to 0 produces a value of A equal to 8.33 which minimizes both L_A^2 and L_A. Substitution of this value into the equations of Table 1 gives the numbers shown. Similar methods are used for all further computations in this article. The computations were carried out on a slide rule, which is rather unusual in these days of computers, and hence there is a possibility of error in the final decimal.

of logrolling. The first is explicit logrolling, most often observed in Congress although it does occur in other situations. It involves individuals who trade their votes on many individual issues to many others for votes on other issues. Under these circumstances, there is no reason why any particular person would be left out. Everyone may trade with anyone else and the result amounts to a peculiar market solution. Since in making the trades, each individual is only attempting to make up a majority coalition, the cost calculations are similar to those informing the agreement we just discussed. Nevertheless, the fact that everyone may get their project means the outcome is different.[6] With just three voters this result would not occur, but with more voters the outcome of this process might well be that all of the harbors would be dredged at a level equivalent to 8 1/3.[7] The individual is in marginal adjustment on those logrolling deals in which he has participated and loses on those in which he has not. L_A, assuming that all three harbors are dredged to the level of expenditure of 8 1/3, is 12.3. This is worse than Mr. A obtains from his simple agreement with Mr. B, but it is certainly much better than he would obtain if no agreements were made at all.

The other type of logrolling is called implicit logrolling and involves political parties or candidates who present "platforms." These platforms, in essence, are complex mixes of different measures. A proposal to dredge two of three harbors would be an example of such a platform. Assuming that this type of logrolling is adopted then the individual *ex ante* has two chances out of three of being a member of the coalition and having his harbor dredged, and one chance in three of having to pay taxes for the dredging of two other harbors. Discounting this out in a simple manner, L_A *ex ante* would be 15.1. Once again, it is much better than a no-logrolling solution. Unfortunately, this type of solution is not mathematically stable, but we will defer discussion of the matter, closely related to Arrow's general impossibility theorem, until the latter part of this article.

So far, we have said nothing about Pareto optimality. If we require unanimity, clearly the bargaining costs would be high, but the economists would normally anticipate that the ultimate outcome, if we disregard the bargaining costs, would be better than the outcomes obtained by partial agreement. The Pareto optimal area is, of course, quite a complex surface running across the three-dimensional space. We can, however, fairly easily compute the value of one particular point on that surface. With our highly symmetric model, side payments would lead to a decision to dredge all three harbors equally at 7 1/7 each and the loss function of that point to Mr. A would be 11.8. *Ex ante* the side payments would cancel out and this is better than any of the other possibilities we have discussed. Needless to say, for an individual who can feel sure that he will be one of a pair of voters who have only their harbors dredged, that outcome would be better than the Pareto optimal outcome.

III. *Five Voters*

The method of calculation we have been describing can readily be applied to any

[6] See James Coleman for a discussion of this point. His article led to comments by R. E. Park and Dennis C. Mueller which, together with a reply by Coleman, are printed in the Dec. 1967 issue of this *Review*, pp. 1300–16.

[7] The simplest way of understanding this problem is to assume that an individual purchases other peoples' votes with his own. There is no obvious reason, if there are more than three voters, why the votes purchased by Mr. A should be the same votes as those purchased by say, Mr. C, although Mr. A's collection of purchases includes Mr. C's vote and Mr. C's collection of purchases includes Mr. A's vote. Mr. A could, for example, make up his majority in a 5-man voting system out of *A*, *B*, and *C*, and Mr. C make up his out of *C*, *D*, and *E*. Mr. E, similarly, might have a majority which consists of C and B as well as himself.

number of dimensions. For example, assume that there are five harbors and five voters (groups and voters) whose loss functions are as shown in Table 2. Once again, if all of the projects for dredging harbors are put up individually and all the individuals vote on them strictly in accordance with their preference on each issue, in each case there will be four votes against and one in favor. The resulting outcome will be the origin of the five-dimensional Cartesian axis system and L_A will be again 22.4. If we assume that three groups of voters, those on harbors A, B, and C, get together to form a majority, they would agree to vote for [7 1/7, 7 1/7, 7 1/7, 0, 0]. This gives L_A equal to 12.0, much better than would be obtained without bargaining. Messrs. D and E, not members of the winning coalition in this case, however, find that the payoff of 25.6 is worse than would have been obtained had logrolling not existed.

TABLE 2

$$L_A^2=5(A-10)^2+B^2+C^2+D^2+E^2$$
$$L_B^2=A^2+5(B-10)^2+C^2+D^2+E^2$$
$$L_C^2=A^2+B^2+5(C-10)^2+D^2+E^2$$
$$L_D^2=A^2+B^2+C^2+5(D-10)^2+E^2$$
$$L_E^2=A^2+B^2+C^2+D^2+5(E-10)^2$$

Using our group of five, however, we can consider a variety of voting rules. Table 3 shows on the left the minimum size of the coalition which is required by various voting rules. The outcome in terms of the amount of dredging in each harbor is shown in the second vertical column, the third column shows the payoff to a member of the winning coalition, and the fourth, the payoff to a man who is left out. In the final column, we show the *ex ante* value of the arrangement for some person who does not know whether he will be a winner or a loser but who assumes his probability of being in the winning coalition is proportional to the number of people required. For comparison purposes, we have put the no-logrolling outcome at the bottom of the table.

The reader way be surprised at the existence of a voting rule permitting a coalition of two to obtain the dredging of their harbors. It is not, however, an unrealistic situation. Most modern democracies use a representative assembly. Under these circumstances, a majority of the voters in a majority of the constituencies may be able to control the outcome. Thus, less than a majority of the voters is necessary. Our two-voter coalition is an example.[8]

It will be noted that the numerical outcomes we have obtained from our simple calculation procedure are in exactly the form which would have been predicted

[8] The proportional representation system used so much on the continent of Europe generally speaking makes it impossible for less than the majority of the voters to have the influence shown. In Anglo-Saxon countries, however, the possibility does exist for the minority of voters obtaining the type of profits shown here.

TABLE 3

Minimum coalition size	Platform	Payoff to member of winning coalition	Payoff to nonmember	*Ex ante* Payoff
2	8 1/3,8 1/3,0,0,0	9.1	27.2	19.9
3	7 1/7,7 1/7, 7 1/7,0,0,	12.0	25.6	17.4
4	6 1/4,6 1/4, 6 1/4, 6 1/4, 0	13.7	25.6	16.1
5	5 5/9,5 5/9, 5 5/9, 5 5/9, 5 5/9	15.0	—	15.0
No log-rolling	0, 0, 0, 0, 0			22.2

from the nonnumerical discussion in Buchanan and Tullock. The costs of coalition formation, of course, must be offset against the numbers in Table 3 to find the optimal voting rule. The point of this model has not been to advance the line of reasoning started in Buchanan and Tullock. Instead it provides a basis for future research by demonstrating that it is possible to obtain their conclusions through a model which differs from the widely used spatial models only by a minor change in parameters.

The outstanding characteristic of the type of issue that normally involves logrolling as opposed to the type of issue that normally does not, is simply that there are groups of voters who feel much more strongly about one particular issue than about others, and that these different groups of voters are arranged roughly in the symmetrical way that we have shown. Needless to say, the perfect symmetry which I have given the model is an aid to calculation, not an effort to describe the real world.

IV. *Other Models*

In order to move from the model we have here to the type of model that was used in Davis and Hinich (1966, 1967, 1968), and Tullock, we may begin by assuming that the individuals favor all of the goods provided to some extent. Suppose for example, that individual A's preferences for the dredging of harbor B is not simply an aversion to taxation for this purpose but that he actually does think it would be nice to have it dredged. Under these circumstances, the center of the disk which now describes his loss function would be moved away from the A-axis a short distance and corresponding computations would indicate that there would be somewhat more dredging of harbor B. This could also lead to the ellipse being shorter and fatter. In the limit, if we continue such operations, we would end with a circle with its center somewhere near the middle of the issue space.

However, we do not have to change our loss functions from disks to spheres in order to obtain approximately the results obtained by the analysis which shows the central policy is dominant. All that is necessary is to relax our extremely strict restrictions upon the shape of individual preferences. We have grouped the individuals in clusters along the axes very strongly favoring certain projects which benefit them and being opposed, mainly because of the tax cost, to individual projects of the same nature in other areas. This is, indeed, a very tight restriction. Unfortunately, it would appear that it is very commonly met in the real world. If we assume that this type of clustering does not occur, then we are back in the world of Davis and Hinich, and Tullock. Thus, we have obtained logrolling essentially out of the spatial model simply by assuming that there are people with the type of preference that we observe in logrolling situations.

So far, however, we have assumed that our function is stable. In actual fact, what we have referred to as explicit logrolling is indeed a stable situation, but, what we have called implicit is not. For example, if we return to the set of equations in Table 1, the platform [10, 10, 0] can be beaten by the platform [10, 0, 0] which can be beaten by [0, 0, 0] which in turn can be beaten by [10, 10, 0]. Further, [10, 10, 0], [0, 10, 10], [10, 0, 10] are all possible winning outcomes. In my *Towards a Mathematics of Politics*, I argued that the instability (implied by the Arrow theorem) in respect to voting was of little real importance. My demonstration, however, depended on the assumption that the number of voters was very much in excess of the number of issue dimensions. When the voters are clustered well out on each of the issue dimensions as they are in our logrolling

model, the proof that I offer ceases to be relevant. In essence, each cluster of voters acts as one voter and the number of such clusters is the same as the number of issue dimensions.

V. *Applications*

In the real world, voting would appear to cover many issues in which the preferences of the individual voters do not have the high degree of structure required for logrolling issues. The classical solution for such a problem for a party wishing to maximize votes would be to seek the middle position on the nonlogrolling issues, and on the logrolling issues, attempt to seize a position which is superior to whatever his opponent has offered. This would lead to sharp changes of policy and great differences between the two parties. We do not observe either of these things in the real world.

It should be noted that a good deal of the logrolling actually done in Congress is on an explicit basis rather than by the parties on an implicit basis. Both the Republican and Democratic congressional candidates from Richmond will be in favor of dredging the harbor. Both will also be against (although not very strongly) dredging other harbors. When they get into the House, the explicit bargaining scheme which is stable will explain their behavior. Unfortunately, there are many types of logrolling which take place at the platform level and hence, the instability problem still remains.

Why do we not see this kind of change in the real world? One possible solution is simply that without the high degree of symmetry which I have imposed upon my model there may be genuinely superior coalitions. Riker (1962) discussed one particular set of conditions under which certain coalitions are "better" than others. There may be many other similar situations.

This solution, however, obviously has its drawbacks and I think we can construct another solution which is both simpler and closer to the real world. In his recent article, Arrow pointed out that "Since the effect of any individual vote is so very small, it does not pay a voter to acquire information unless his stake in the issue is enormously greater than the cost of the information." These theoretical considerations which indicate that people should not bother to become informed about politics can be matched with empirical data which seems to indicate that they do not, in fact, know much about politics. If we assume that individuals will only make an effort to find out about policies when the effect on them is greater than a certain amount, then the individual would normally know, at least, something about what we might call public interest issues, such as police and national defense and also something about those logrolling issues which particularly concern him, i.e., those upon which his feelings are intense. However, he would not know anything about those logrolling issues which did not greatly affect him.

All the inhabitants of Richmond would know about the James River dredging project but few of them would know about the dredging of the river to Tulsa, Oklahoma. Under these circumstances, a political party making up its platform would assume that different voters have somewhat different information positions. In the extreme, the voter in Richmond would respond to a political world in which he saw general issues and the dredging of the James. In this area of his information there will be no possibility of strict logrolling and hence both political parties would choose approximately the center of this issue space. With our assumptions of voter ignorance of other issues, this would involve dredging the James to the level of 10. With many voters in Richmond, the point chosen would be the median of *their* pref-

erences. The party also assumes similar positions with respect to other electorates which have different fields of knowledge and preferences. The outcome would involve logrolling in a sense that the individual groups would be given special treatment but would depend upon the ignorance of the voter with respect to the logrolling "payments" to other parties. Whether voters actually *are* this ignorant is something which can be questioned. Certainly they are opposed to taxes in general, and are aware of the fact that other people's projects, in one way or another, contribute to the tax load. The empirical investigations which do show appalling voter ignorance have never been addressed to this specific problem. Further empirical research would appear to be called for.

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